Enrollment No. _____________

Shree Manibhai Virani and Smt. Navalben Virani Science College (Autonomous), Rajkot
Affiliated to Saurashtra University, Rajkot

SEMESTER END EXAMINATION APRIL - 2018

M.Sc. Mathematics

16PMTCC17 – FUNCTIONAL ANALYSIS

Duration of Exam – 3 hrs ***Semester – IV*** ***Max. Marks – 70***

***Part A** (5X2= 10 marks)*

Answer **ALL** questions

1. Explain complete space with an example.
2. Define: $B(X,Y)$.
3. Define: Hilbert spaces.
4. Define: Hilbert adjoint operator.
5. Are every finite dimensional normed space is reflexive? Why?

***Part B** (5X5= 25 marks)*

Answer **ALL** questions

6a. State and prove Riesz lemma.

OR

6b. Define equivalent norms. Prove that on a finite dimensional vector space X any norm $\|.\|$ is equivalent to any other norm $\|.\|_0$.

7a. Show that the dual space of l^1 is l^∞.

OR

7b. Define a linear operator. Show that for a linear operator T, the range $\Re(T)$ is a vector space.

8a. Let X be an inner product space over K and $x, y \in X$ be such that x is orthogonal to y, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. If X be a real inner product space then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ implies x is orthogonal to y.

OR

8b. In an inner product space X, $\langle x,u\rangle = \langle x,v\rangle, \forall\, x$, show that $u = v$.
Let $x \neq 0, y \neq 0$, if x is orthogonal to y, then show that $\{x, y\}$ is linearly independent.

9a. State and prove Hahn Banach Theorem for normed spaces.

OR

9b. Let H_1 and H_2 be Hilbert spaces. $T : H_1 \to H_2$ be the bounded linear operators. Then show that

i) $\langle T^* y, x\rangle = \langle y, Tx\rangle, \forall\, x \in H_1, y \in H_2$

ii) $\|T^*T\| = \|TT^*\| = \|T\|^2$

10a. Let X be a normed space and $\{x_n\}$ be a sequence in X. Then show that the sequence $\{\|x_n\|\}$ is bounded.

OR

10b. State and prove Uniform boundedness theorem.

<u>Part C</u> (5X7= 35 marks)

Answer **<u>ALL</u>** questions

11a. Show that the Banach space $L^P[a,b]$ is the completion of the normed space which consists of all continuous real valued functions on $[a,b]$.

OR

11b. In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

12a. Let T be a linear operator. Then prove that,

i) the range R(T) is a vector space

ii) if $\dim D(T) < \infty$ and T^{-1} exists, then $\dim R(T) = \dim D(T)$

OR

12b. Prove that the dual space X' of a normed space X is a Banach space.

13a. Orthonormalize the first three terms of the sequence $(x_0, x_1, x_2, ...)$ where $x_j(t) = t^j$ on the interval [-1,1] where $\langle x, y \rangle = \int_{-1}^{1} x(t)y(t)dt$.

OR

13b. An orthonormal set M in a Hilbert space H is a total in H if and only if for all fixed $x \in H$, the Parseval relation $\sum_k |\langle x, e_k \rangle|^2 = \|x\|^2$ holds.

14a. Let the operators $U, V : H \to H$ be Unitary, where H is a Hilbert space. Then prove that

i) U is isometric.

ii) UV is Unitary.

iii) U is normal.

OR

14b. Prove: Let X be a normed space and $x_o \neq 0$ be any element of X. Thus there exist bounded linear functional $\tilde{f}$ on X such that $\|\tilde{f}\| = 1$, $\tilde{f}(x_o) = \|x_o\|$. For every x in normed space X, $\|x\| = \sup_{f \in X'} \frac{|f(x)|}{\|f\|}$.

15a. For a normed space X, prove that a strong convergence in X implies weak convergence. Is the converse true? Justify your answer.

OR

15b. State and prove closed graph theorem